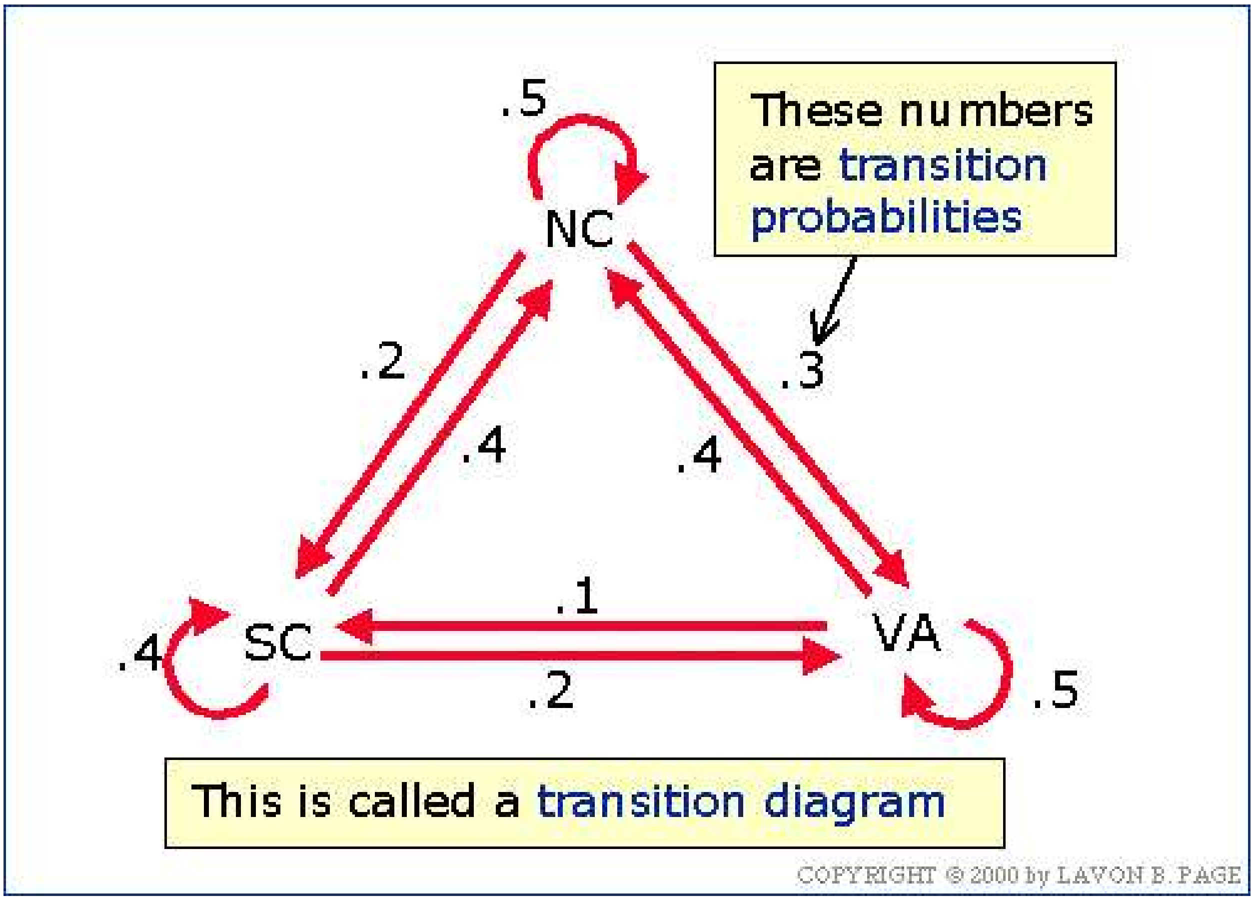
I found these examples on the Internet at http://www.math.ncsu.edu/ma114/, the North Carolina State University website. I received permission from Lavon B. Page, the author, to use the examples.

Example 1: John’s Truck Rental

John’s Truck Rental does business in North Carolina, South Carolina, and Virginia. As with most rental agencies, customers may return the vehicle that they have rented at any one of the company’s franchises throughout the three-state area. In order to keep track of the movement of its vehicles, the company has accumulated the following data: 50% of the trucks rented in North Carolina are returned to North Carolina locations, 30% are dropped off in Virginia, and 20% in South Carolina. Of those rented in South Carolina, 40% are returned to South Carolina, 40% are returned in North Carolina, and 20% in Virginia. Of the trucks rented in Virginia, 50% are returned to Virginia, 40% in North Carolina, and 10% in South Carolina.



The transition probabilities can be put into a matrix called the <u>transition matrix</u>.

$$
\begin{array}{c} \\ NC \\ SC \\ VA \end{array}
\begin{array}{c}
\begin{array}{ccc} NC & SC & VA \end{array} \\
\begin{bmatrix} .5 & .2 & .3 \\ .4 & .4 & .2 \\ .4 & .1 & .5 \end{bmatrix}
\end{array}
$$

The row headers tell us the state from which the truck is rented, and the column headers tell us the state to which the truck is returned.

Each row in the matrix must add up to 1.

What fraction of the time does a truck spend in each of the 3 states?

We have to solve for

$$[x \quad y \quad z]\begin{bmatrix} .5 & .2 & .3 \\ .4 & .4 & .2 \\ .4 & .1 & .5 \end{bmatrix} = [x \quad y \quad z]$$

remembering that $x + y + z = 1$.

So the system of equations is

$$.5x + .4y + .4z = x$$

$$.2x + .4y + .1z = y$$

$$.3x + .2y + .5z = z$$

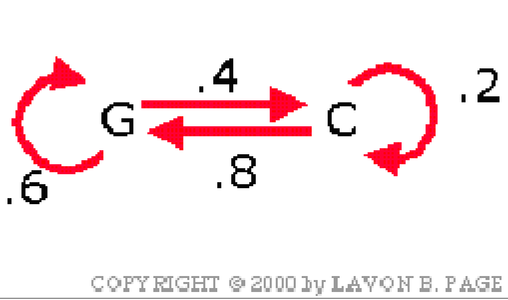
$$x + y + z = 1$$

The equations in standard form:	The matrix to solve:	The solution:
$\begin{array}{rrrcl} x & +y & +z & = & 1 \\ -.5x & +.4y & +.4z & = & 0 \\ .2x & -.6y & +.1z & = & 0 \\ .3x & +.2y & -.5z & = & 0 \end{array}$	$\left[\begin{array}{rrr\|r} 1 & 1 & 1 & 1 \\ -.5 & .4 & .4 & 0 \\ .2 & -.6 & .1 & 0 \\ .3 & .2 & -.5 & 0 \end{array}\right]$	$\left[\begin{array}{rrr\|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

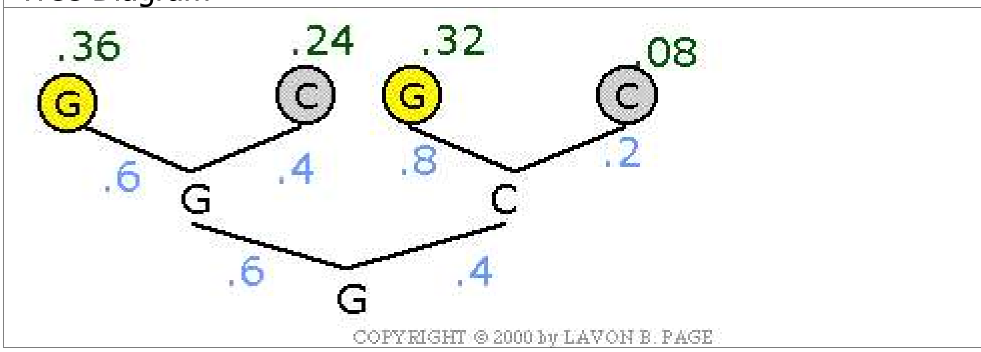
Example 2: John and his Classes

If John goes to class one day, there is a 60% chance he will go the next day. If he cuts class one day, there is an 80% chance he will go the next day. The class meets every day.

G = Goes to Class
C = Cuts Class

Transition Diagram	Transition Matrix
	$\begin{array}{cc} & \begin{array}{cc} G & C \end{array} \\ \begin{array}{c} G \\ C \end{array} & \begin{bmatrix} .6 & .4 \\ .8 & .2 \end{bmatrix} \end{array}$ COPYRIGHT © 2000 by LAVON B. PAGE

If John goes to class on Monday, what is the probability he will go to class on Wednesday?

Tree Diagram


The bottom row represents going to class on Monday.
The middle row represents the probabilities of going to or cutting class on Tuesday.
The top row represents the probabilities for Wednesday, depending on what he does Tuesday.

Adding together the probabilities for the desired end result, going to class on Wednesday, we get:

$$0.36 + 0.32 = 0.68$$

We can get the same result if we multiply the transition matrix by itself:

$$T \times T = \begin{bmatrix} .6 & .4 \\ .8 & .2 \end{bmatrix} \begin{bmatrix} .6 & .4 \\ .8 & .2 \end{bmatrix} = \begin{bmatrix} .68 & .32 \\ .64 & .36 \end{bmatrix}$$

$$T^2 = \begin{array}{cc} & \begin{array}{cc} G & C \end{array} \\ \begin{array}{c} G \\ C \end{array} & \begin{bmatrix} .68 & .32 \\ .64 & .36 \end{bmatrix} \end{array}$$

The entries are the two-step transition probabilities.

Example 3: Smokers and Non-Smokers

Suppose we are tracking smokers and non-smokers:

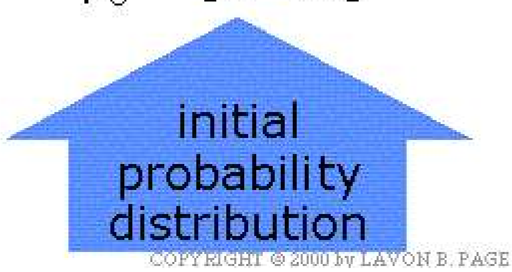
S = Smokers
N = Non-smokers

Each year, 10% of smokers stop smoking, and 5% of non-smokers start smoking.

If initially 40% of the population smokes, track the trend over 4 years.

$$T = \begin{array}{c} \\ S \\ N \end{array}\!\!\begin{array}{c} \begin{array}{cc} S & N \end{array} \\ \begin{bmatrix} .9 & .1 \\ .05 & .95 \end{bmatrix} \end{array}$$

$$p_0 = [.4 \quad .6]$$



Multiply the initial probability matrix p_0 by the transition matrix T to get the first state matrix p_1 (at the end of year 1).

$$p_1 = p_0 T = [0.4 \quad 0.6]\begin{bmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{bmatrix} = [0.39 \quad 0.61]$$

Multiply the first state matrix p_1 by the transition matrix T to get the second state matrix p_2 (at the end of year 2).

$$p_2 = p_1 T = p_0 T^2 = [0.4 \quad 0.6]\begin{bmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{bmatrix}^2 = [0.3815 \quad 0.6185]$$

Multiply the second state matrix p_2 by the transition matrix T to get the third state matrix p_3 (at the end of year 3).

$$p_3 = p_2 T = p_0 T^3 = [0.4 \quad 0.6]\begin{bmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{bmatrix}^3 \approx [0.3743 \quad 0.6257]$$

Multiply the third state matrix p_3 by the transition matrix T to get the fourth state matrix p_4 (at the end of year 4).

$$p_4 = p_3 T = p_0 T^4 = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.05 & 0.95 \end{bmatrix}^4 \approx \begin{bmatrix} 0.3681 & 0.6319 \end{bmatrix}$$

Example 4: Frisbee Game

Bob, Alice and Carol are playing Frisbee. Bob always throws to Alice and Alice always throws to Carol. Carol throws to Bob 2/3 of the time and to Alice 1/3 of the time. In the long run, what percentage of the time do each of the players have the Frisbee?

$$T = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} \end{array}$$

We must solve the matrix equation

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1/3 & 2/3 & 0 \end{bmatrix} = \begin{bmatrix} x & y & z \end{bmatrix}$$

and use the fact that $x + y + z = 1$

The equations to solve are:

1. $y + \frac{1}{3}z = x$
2. $\frac{2}{3}z = y$
3. $x = z$
4. $x + y + z = 1$

Because $x = z$:

1. $y = \frac{2}{3}z$
2. $\frac{2}{3}z = y$
4. $2z + y = 1$

Because $y = \frac{2}{3}z$

4. $\frac{8}{3}z = 1$

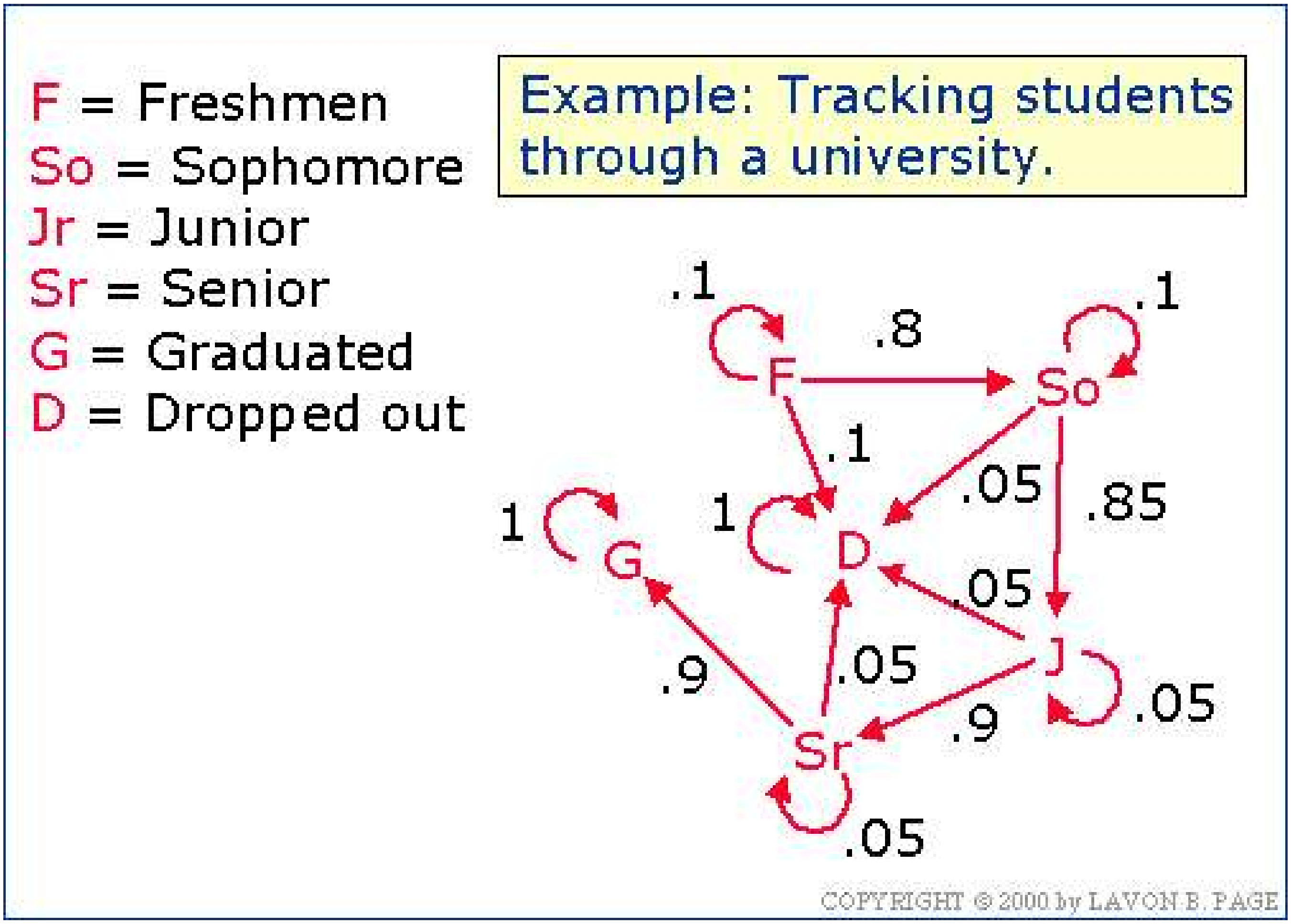
Therefore:

$z = \frac{3}{8}$; $x = \frac{3}{8}$; $y = \frac{1}{4}$

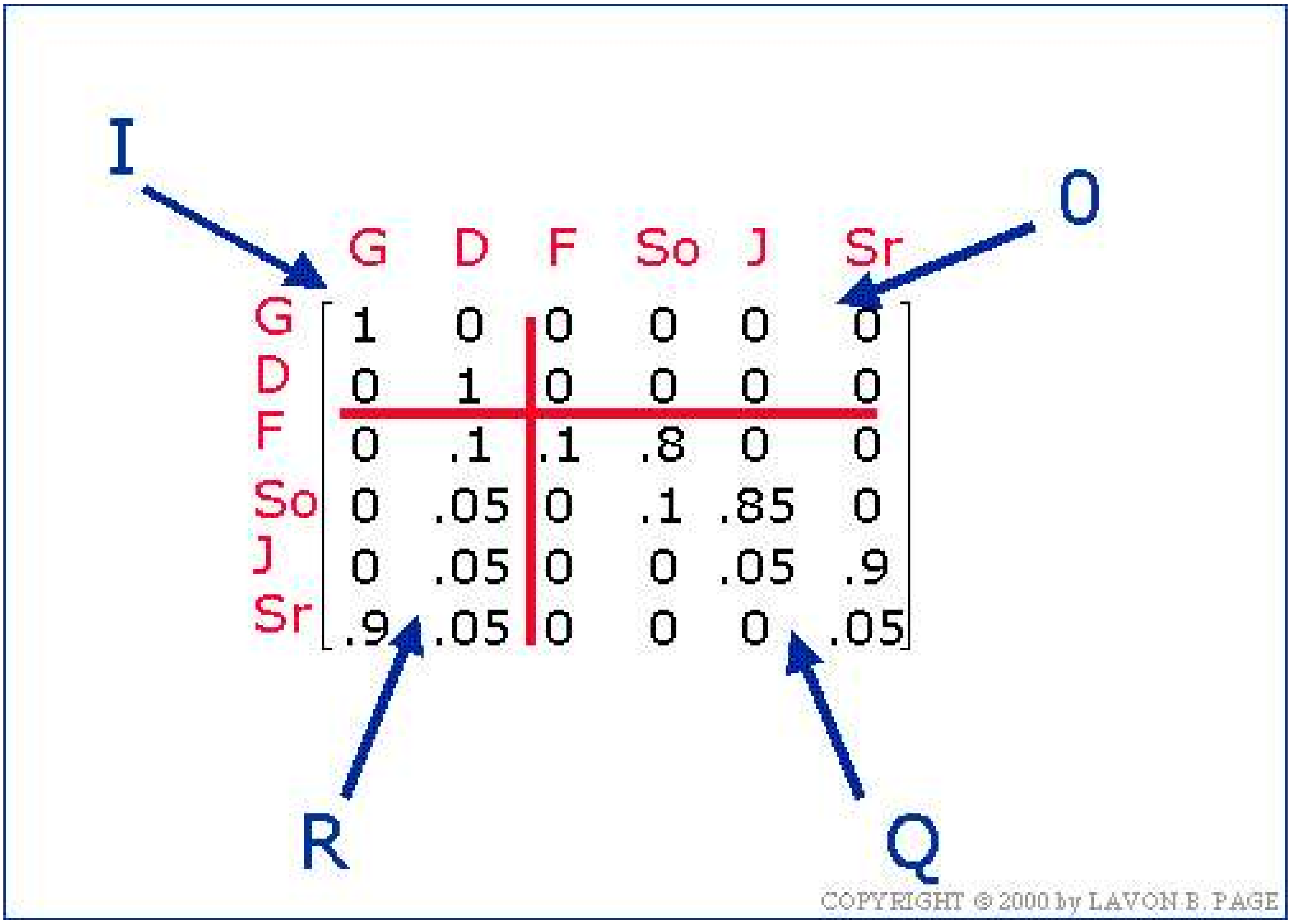
So:

Alice has the Frisbee 3/8 of the time.
Bob has the Frisbee 1/4 of the time.
Carol has the Frisbee 3/8 of the time.

Example 5: Tracking Students



Here is the transition matrix in standard form:



The matrix below is T^3 (approximately).

$$
\begin{array}{c}
 \\ G \\ D \\ F \\ So \\ J \\ Sr
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \quad G \quad & \quad D \quad & \quad F \quad & \quad So \quad & \quad J \quad & \quad Sr \quad \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & .193 & .001 & .024 & .170 & .612 \\
.689 & .142 & 0 & .001 & .0149 & .153 \\
.891 & .102 & 0 & 0 & .000125 & .00675 \\
.947 & .0526 & 0 & 0 & 0 & .000125
\end{bmatrix}
\end{array}
$$

Probability that a freshman will be a senior after 3 years in school.

The matrix below is T^3 (approximately).

$$
\begin{array}{c}
\\ G \\ D \\ F \\ So \\ J \\ Sr
\end{array}
\begin{array}{c}
\begin{array}{cccccc} G & D & F & So & J & Sr \end{array} \\
\left[\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & .193 & .001 & .024 & .170 & .612 \\
.689 & .142 & 0 & .001 & .0149 & .153 \\
.891 & .102 & 0 & 0 & .000125 & .00675 \\
.947 & .0526 & 0 & 0 & 0 & .000125
\end{array}\right]
\end{array}
$$

Of special interest is this part of the matrix, because it indicates the probabilities of being in each of the absorbing states after 3 years.

What are the long-term probabilities for entering each absorbing state?

Solving $B = (I - Q)^{-1} R$, we get:

$$B = NR = \begin{array}{c} \\ F \\ So \\ J \\ Sr \end{array} \begin{array}{c} \begin{array}{cc} G & \quad D \end{array} \\ \begin{bmatrix} .753 & .247 \\ .848 & .152 \\ .898 & .102 \\ .947 & .053 \end{bmatrix} \end{array}$$

Label the rows with the non-absorbing states and the columns with the absorbing states.

Let’s look at matrix N, which is given by $(I-Q)^{-1}$.

$$
\begin{array}{c|cccc|c}
 & F & So & J & Sr & \\
\hline
F & 1.111 & .988 & .884 & .837 & 3.819 \\
So & 0 & 1.111 & .994 & .942 & 3.047 \\
J & 0 & 0 & 1.053 & .997 & 2.05 \\
Sr & 0 & 0 & 0 & 1.053 & 1.053
\end{array}
$$

Each entry represents the average amount of time a student will spend at a certain level, depending on what level they entered the college.

If a student enters as a freshman, he/she will spend an average of 1.111 years as a freshman, 0.988 years as a sophomore, 0.884 years as a junior, and 0.837 years as a senior. Altogether, the student will spend an average of 3.819 years in college.

The fact that the 4-year average is low indicates that students are likely finishing early or dropping out, rather than taking more time.