Introduction

In this chapter, we are going to talk about physical **systems** and their **states**.

A physical system is anything with a set of observable properties:

- The value of a stock on the NYSE.
- The party affiliation of a voting precinct.

The state of the physical system is the value of the observable properties:

- Stock value has increased, decreased, or remained the same.
- The voting precinct might be Democratic, Republican, or Other.

When a system evolves from one state to another by chance, the progression is called a stochastic process. Through observation over time, we can assign probabilities of evolving from one state to the next.

Transition and State Matrices

A toothpaste company markets a product (brand A) that currently has 10% of the market. A consumer is in one of two states:

A	Uses brand A	0.10
A′	Uses another brand	0.90

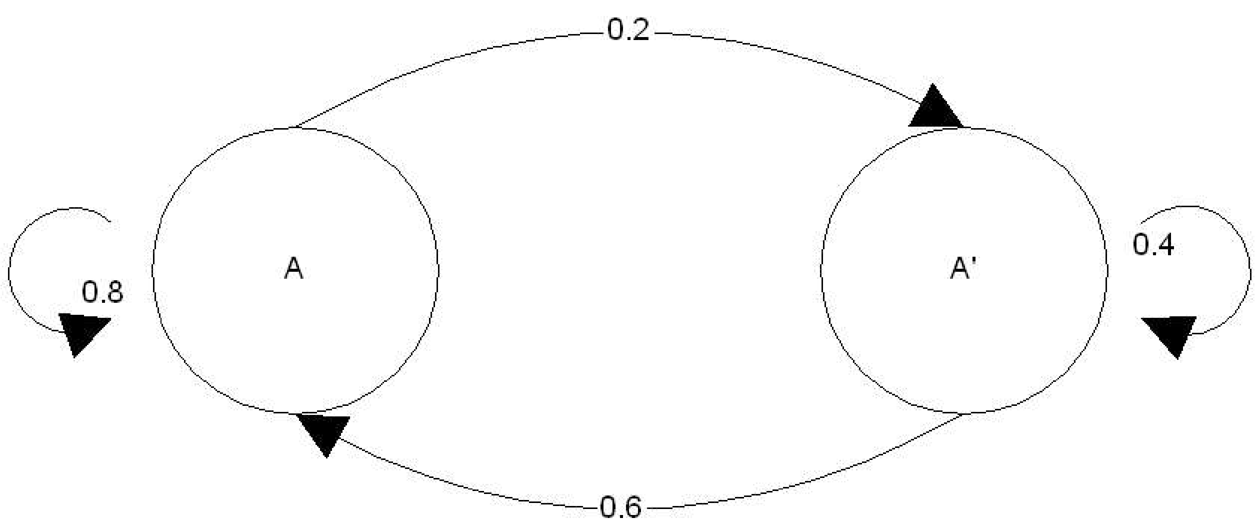
The **initial-state distribution matrix** is:

$$S_0 = \begin{array}{c} \begin{array}{cc} A & A' \end{array} \\ [\ \begin{array}{cc} 0.1 & 0.9 \end{array}\ ] \end{array}$$ (note that the row adds to 1.0)

The company hires a market research firm to estimate how much of the market it might acquire if they launch an aggressive sales campaign. The results:

- 80% of consumers using brand A will buy it again.
- 60% of consumers not using brand A will switch to brand A

This can be displayed in a **transition diagram**:



This can also be displayed in a **transition probability matrix**, where the rows are the current state and the columns are the next state:

Next State

$$\text{Current State: } \begin{array}{c c} & \begin{array}{cc} A & A' \end{array} \\ \begin{array}{c} A \\ A' \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} \end{array} = P$$ (note that each row and column add to 1)

The first-state matrix is obtained by multiplying the initial state matrix by the transition probability matrix:

$$S_1 = S_0 P = \begin{bmatrix} 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.62 & 0.38 \end{bmatrix}$$

This tells us that, after the marketing campaign, 62% of the consumers will be using brand A.

Multiplication of matrices is a matter of calculating inner products:

1. Pair up the elements from a row of the first matrix with a column of the second matrix (this is why number of columns has to match number of rows).
2. Multiply each pair together.
3. Add all the results together.

Each entry in the resultant matrix is an inner product of the two original matrices.

1. The inner product of the **1st** row and **1st** column is the entry in row **1** column **1**.
2. The inner product of the **1st** row and **2nd** column is the entry in row **1** column **2**.
3. The inner product of the **2nd** row and **1st** column is the entry in row **2** column **1**.
4. The inner product of the **2nd** row and **2nd** column is the entry in row **2** column **2**.

We can repeat the process and find the second-state matrix – we multiply the first-state matrix by the transition probability matrix:

$$S_2 = S_1 P = \begin{bmatrix} 0.62 & 0.38 \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.6 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.724 & 0.276 \end{bmatrix}$$

The process may be repeated as many times as needed.

The series of state matrices is known as a Markov Chain.

Powers of Transition Matrices

As we calculate each state, all we are doing is multiplying by P each time:

$$S_1 = S_0 P$$
$$S_2 = S_1 P = S_0 P^2$$
$$S_3 = S_2 P = S_0 P^3$$

$$S_k = S_0 P^k$$

These calculations can very easily be done in a calculator with matrix capabilities.